Recursive formula for the “components” of a hyper-cube[*]

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Starting from considerations regarding the “components” of cubes in various dimensions we find a recursive formula $H_n^m = 2 \cdot H_{n-1}^m + H_{n-1}^{m-1}$ *which, incidentally, when generalized gives Euler's formula up to 3-d space and also establishes an analog of the formula for other spaces as well.*

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Euler's formula, $V - E + F = 2$, gives a relationship between the vertices edges and faces of polyhedra. A cube has, in fact, 8 vertices, 12 edges and 6 faces. Let us call vertices, edges and faces “components” of the cube. Now a

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[*] typeset by LaTeX

square has 4 edges, 4 vertices and one face (which is the square itself). The components of the line segment are two vertices and one edge, the line itself.

We may imagine a cube being generated by moving a square orthogonally with respect to the plane containing the square. The square, in turn, being produced by "dragging" a line segment (its length is one side of the square) along the plane a distance equal to its length. We can say the square has been "generated" by the line segment. But the line segment itself may be considered as the one dimensional analog of the square, "generated" in turn by a point.

Going in the other direction, towards higher dimensions, with the cube we can generate a 4-dimensional hyper-cube, and so on.

One may then ask how many components make it up, that is, how many faces, vertices, edges and cubes (its analog of the 3-d faces) can be counted in it? For that matter, given an n^{th} dimensional cube, how many components is it made up of?

In order to answer this let us first devise a suitable notation.

Let us designate a component by H, with the superscript indicating which component we are dealing with, and the subscript indicating in which dimension. The integer value of H^i_j then indicates the i^{th} component in j^{th} space.

So a vertex (point) in a 1-dimensional space will be H_0^0, whereas, for example, a cube will contain H_3^0 vertices. The number of edges (i.e. lines) for a given polygon in n^{th}-dimensional space will be H_n^1, a face will be H_n^2, a cube H_n^3 and so on. Obviously this will allow us to indicate as many dimensions as necessary, a 7^{th}-dimensional hyper-cube will contain $H_7^7, H_7^6, H_7^5 \dots H_7^0$ components.

Let me introduce a recursive formula which will specify how many of a given component there are:

$$\mathbf{H_n^m = 2 \cdot H_{n-1}^m + H_{n-1}^{m-1}} \tag{1}$$

Given $H_0^0 = 1$, we will define $H_n^{-1} = 0\ \forall n$, thus giving the end values for the recursive formula. We should note that H may only assume non-negative values, for $m \leq n$, otherwise it will be considered zero.[1]

Let us build up some concrete examples

For 0-dimensions we only have a point, and we have formally only one vertex, the point itself:

$$H_0^0 = 1$$

Given a line, we may consider a segment the 1-dimensional analog to the

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[1] That is, for $m \geq n \Rightarrow H_n^m = 0$.

2-dimensional square, and the formula gives us, the number of vertices:

$$H_1^0 = 2 \cdot H_0^0 + H_{-1}^{-1} = 2 \cdot 1 + 0 = 2$$

and the number of edges (H^1):

$$H_1^1 = 2 \cdot H_0^1 + H_0^0 = 0 + 1 = 1$$

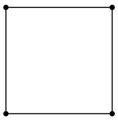
At this point we assert that, $H_n^n = 1$. The proof is by induction, since $H_0^0 = 1$, assuming it true for any particular n it follows that it is true for $n + 1$.

$$H_{n+1}^{n+1} = 2 \cdot H_n^{n+1} + H_n^n = 2 \cdot 0 + 1 = 1$$

That this should be so is intuitively clear, since it is the n^{th} dimensional figure itself in n space that is being counted once.

It is also trivially true that $H_n^0 = 2 \cdot H_{n-1}^0$. Again, it stands to reason, geometrically, that the number of vertices doubles as we "*drag*" an n-dimensional figure to create the $(n+1)^{th}$-dimensional one.

That $H_n^1 = 2 \cdot H_{n-1}^1 + H_{n-1}^0$ also stands to reason, since the number of edges will be twice that of the figure in $(n-1)$-space (its initial and final positions) plus the new edges created by the vertices being "dragged"through n-space.

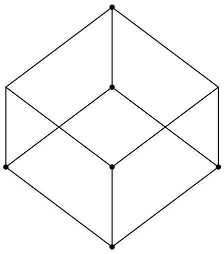


Now looking at a square, we have, for the vertices, following the assertion made above:

$$H_2^0 = 2 \cdot H_1^0 = 2 \cdot 2 = 4$$

and for the edges:

$$H_2^1 = 2 \cdot H_1^1 + H_1^0 = 2 \cdot 1 + 2 = 4$$



To find the number of vertices H_3^0 in a cube we find $2 \cdot H_2^0 = 8$ and the first two interesting results, the number of edges:

$$H_3^1 = 2 \cdot H_2^1 + H_2^0 = 2 \cdot 4 + 4 = 12$$

and number of faces:

$$H_3^2 = 2 \cdot H_2^2 + H_2^1 = 2 \cdot 1 + 4 = 6$$

Continuing some of the generalizations we get the following closed formulas,

$$\begin{aligned}
H_n^0 &= 2 \cdot H_{n-1}^0 \\
&= 2 \cdot \left(2 \cdot H_{n-2}^0\right) \\
&= \underbrace{2 \cdot 2 \cdots 2}_{n} \cdot H_{n-n}^0 \\
&= 2 \cdot 2 \cdots 2 \cdot H_0^0 \\
&= 2^n
\end{aligned}$$

we also have

$$\begin{aligned}
H_{n+1}^n &= 2 \cdot H_n^n + H_n^{n-1} \\
&= 2 \cdot H_n^n + \left(2 \cdot H_{n-1}^{n-1} + H_{n-1}^{n-2}\right) \\
&= 2 \cdot H_n^n + 2 \cdot H_{n-1}^{n-1} + \cdots + 2 \cdot H_2^2 \left(2 \cdot H_1^1 + H_1^0\right) \\
&= \underbrace{2 + 2 + \cdots + 2 + (2 + 2)}_{n+1} \\
&= 2(n+1)
\end{aligned}$$

as well as:

$$\begin{aligned}
H_n^1 &= 2 \cdot H_{n-1}^1 + H_{n-1}^0 \\
&= 2 \cdot H_{n-1}^1 + 2^{n-1} \\
&= 2 \cdot \Big(2 \cdot H_{n-2}^1 + H_{n-2}^0\Big) + 2^{n-1} \\
&= 2 \cdot \Big(2 \cdot H_{n-2}^1\Big) + 2 \cdot 2^{n-2} + 2^{n-1} \\
&= 2 \cdot \Big(2 \cdot H_{n-2}^1\Big) + 2 \cdot 2^{n-1} \\
&= \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n-1} \cdot H_1^1 + (n-1) \cdot 2^{n-1} \\
&= n \cdot 2^{n-1}
\end{aligned}$$

Carrying on some further generalizations we get the following closed formulas:

$$\begin{aligned}
H_n^2 &= 2 \cdot H_{n-1}^2 + H_{n-1}^1 \\
&= 2 \cdot H_{n-1}^2 + (n-1) \cdot 2^{n-2} \\
&= 2 \cdot \Big(H_{n-2}^2 + H_{n-2}^1\Big) + (n-1)2^{n-2} \\
&\ldots \\
&= \tfrac{(n \cdot (n-1))}{2} 2^{n-2} = n \cdot (n-1)2^{n-3}
\end{aligned}$$

For H_n^3 we proceed along the same lines as above, it is just a bit more tedious.

$$\begin{aligned} H_n^3 &= 2 \cdot H_{n-1}^3 + H_{n-1}^2 \\ &= 2 \cdot H_{n-1}^3 + (n-1)(n-2) \cdot 2^{n-4} \\ &= 2 \cdot \left(H_{n-2}^3 + H_{n-2}^2\right) + (n-1)(n-2)2^{n-4} \\ &\dots \\ &= 2^{n-3} \cdot \left[(n^3 - 3n^2 + 2n + 9)\,\tfrac{1}{3}\right] \end{aligned}$$

These may be proved by induction.

To summarize the closed formulae,

$$H_n^n = 1$$

$$H_n^0 = 2^n$$

$$H_{n+1}^n = 2 \cdot (n+1)$$

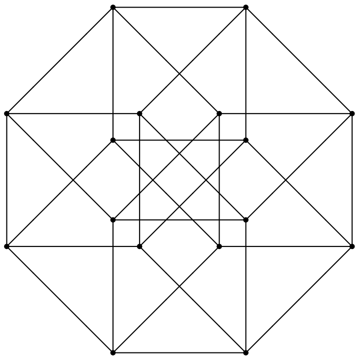
$$H_n^1 = n \cdot 2^{n-1}$$

$$H_n^2 = n \cdot (n-1) \cdot 2^{n-3}$$

$$H_n^3 = 2^{n-3} \cdot \left[(n^3 - 3n^2 + 2n + 9)\,\tfrac{1}{3}\right]$$

It may be interesting to see these in Euler's formula, but modifying it somewhat,[2] applied to a cube (or rectangular prism):

$$\begin{aligned} V - E + F - C &= H_3^0 - H_3^1 + H_3^2 - H_3^3 \\ &= 2^3 - 3 \cdot 2^{(3-1)} + 2 \cdot 3 - 1 \\ &= 8 - 12 + 6 - 1 = 1 \end{aligned}$$



For the *hyper-cube*, H_4^4, we have $H_4^0 = 2 \cdot H_3^0 = 16$ vertices, and the number of edges:

$$H_4^1 = 2 \cdot H_3^1 + H_3^0 = 2 \cdot 12 + 8 = 32$$

number of faces that make up a hyper-cube:

$$H_4^2 = 2 \cdot H_3^2 + H_3^1 = 2 \cdot 6 + 12 = 24$$

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[2] The modification is to include the 3-d figure itself, denoted by "C". The reason why the formula has been modified will become clear shortly.

and number of $3d$ components or cubes:

$$H_4^3 = 2 \cdot H_3^3 + H_3^2 = 2 \cdot 1 + 6 = 8$$

again applying the (analog) of Euler's formula:

$$\begin{aligned} V - E + F - C + H &= H_4^0 - H_4^1 + H_4^2 - H_4^3 + H_4^4 \\ &= 16 - 32 + 24 - 8 + 1 = 1 \end{aligned}$$

What is more interesting, generalizing Euler's formula using this notation we get:

$$\sum_{i=0}^{n} (-1)^i \cdot H_n^i = +H_n^0 - H_n^1 + H_n^2 - H_n^3 + H_n^4 - H_n^5 \cdots \pm H_n^n$$

we have, for the first few spaces $\mathcal{E}^n$

1. $\mathcal{E}^0 \quad H_0^0 = 1$
2. $\mathcal{E}^1 \quad H_1^0 - H_1^1 = 1$
3. $\mathcal{E}^2 \quad H_2^0 - H_2^1 + H_2^2 = 1$
4. $\mathcal{E}^3 \quad H_3^0 - H_3^1 + H_3^2 - H_3^3 = 1$
5. $\mathcal{E}^4 \quad H_4^0 - H_4^1 + H_4^2 - H_4^3 + H_4^4 = 1$

For $\mathcal{E}^0$ and $\mathcal{E}^1$ the formulae are trivially true, and it is quite easy to show it for $\mathcal{E}^2$ as well.[3] In $\mathcal{E}^3$ it is a trivial extension of Euler's formula, so that this is also true for all polygons.

The 4^{th} dimensional analog of the tetrahedron, the smallest regular solid in $\mathcal{E}_4$ has 5 vertices, 10 edges, 10 faces and 5 (3-d) tetrahedrons. We can see that it too satisfies the general formula.

In $\mathcal{E}^4$ the proof for all hypercubes follows along the line of a proof for Euler's formula (at least as given in Courant's "*What is Mathematics?*"), $(H_n^n - 1) - (H_n^{n-1} - 1) + \ldots (H_n^0) = 1$ leaves the value unvaried since the signs always alternate. We can continue the reasoning with the remaining components until we have reduced it to the identity $1 = 1$ and this can be extended to $\mathcal{E}^1$ for any n hypercubes.

Now it remains to be proved for figures other than hyper-cubes, and my conjecture is that

for any n-dimensional space, for any simple convex polytope:

$$\sum_{i=0}^{n} (-1)^i \cdot H_n^i = 1 \tag{2}$$

for which I am presently working on a proof.

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[3] Every plane polygon (including concave polygons) has one side for each vertex, and thus $H_2^0 = H_2^1$.