1 Fourier Series

Considering a function, $f(\theta)$, we seek an expansion for it of the form

$$f(\theta) = \frac{A_0}{2} + \sum \left(A_n \cos n\theta + B_n \sin n\theta\right) \tag{1}$$

where the coefficients can be found by multiplying both sides by $\cos n\theta$ (or by $\sin n\theta$) and integratinf from 0 to 2π

In general we call a Fourier series corresponding to function $f(x)$ of period L

$$f(x) = \frac{a_o}{2} + \sum_n \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right) \tag{2}$$

where the Fourier coefficients are

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \tag{3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \tag{4}$$

and the term a_o, which is the mean of $f(x)$ over a period, is

$$\frac{a_o}{2} = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \tag{5}$$

while for periodic functions having period $2L$, $f(x) = f(x+2L)$ we can write

$$a_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin \frac{n\pi x}{L}\, dx$$

2 Miscellaneous results

We can prove that:

$$\int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx = \int_{-L}^{L} \sin \frac{n\pi x}{L}\, dx = 0$$

and that

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

as well as

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$$

3 Parseval's identity

Parseval's identity, if a_n, b_n are the Fourier coefficients corresponding to $f(x)$ satisfying Dirichlet conditions, is

$$\frac{1}{L}\int_{-L}^{L} (f(x))^2\,dx = \frac{a_o^2}{2} + \sum_n \left(a_n^2 + b_n^2\right) \tag{6}$$

4 Fourier complex representation

Given the relations:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

(supposing Dirichlet conditions hold) we have the complex notation for Fourier series:

$$f(\theta) = \sum_n c_n\,e^{in\theta} \tag{7}$$

and we observe that

$$\int_0^L e^{-2i\pi mx/L}e^{2i\pi nx/L}\,dx = L\delta_{mn}$$

therefore

$$\frac{1}{L}f(x)\int_0^L e^{-2i\pi mx/L}\,dx = \sum_{n=-\infty}^{\infty} c_n L\delta_{mn} = c_m$$

or more generally

$$f(x) = \sum_n c_n\,e^{in\pi x/L} \tag{8}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx \tag{9}$$

5 Double (sine) Fourier series

We have

$$f(x,y)=\sum_m\sum_n B_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi x}{L_2} \tag{10}$$

where

$$B_{mn}=\frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} f(x,y)\sin\frac{m\pi x}{L_1}\sin\frac{n\pi x}{L_2} \tag{11}$$

6 Legendre polynomials

Using the orthogonalization theorem:

$$y_{m+1}=x_{m+1}-\sum_j\frac{(x_{m+1},y_j)}{(y_j,y_j)}y_j \tag{12}$$

where the inner product in the linear space of polynomials is

$$(x,y)=\int_{-1}^{1}x(t)y(t)\;dt$$

and given the sequence $x_n(t)=t^n$ we obtain $x_0=1$, $y_0=1$, $x_1=t$, $y_1=t$ and $y_3(t)=t^3-\frac{3}{5}t$, $y_4(t)=t^4-\frac{6}{7}t^2+\frac{3}{35}$, $y_5(t)=t^5-\frac{10}{9}t^3+\frac{5}{21}t$

$$(y_0,y_0)=\int_{-1}^{1}=2 \qquad (x_1,y_0)=\int_{-1}^{1}=0$$

$$(x_2,y_0)=\frac{2}{3} \qquad (x_2,y_1)=0$$

$$(y_1,y_1)=\frac{2}{3}$$

In general we have

$$y_n(t)=\frac{n!}{(2n)!}\frac{d^n(t^2-1)^n}{dt^n}$$

and Legendre polynomials are defined by the formula

$$P_n(t)=y_n(t)\frac{(2n)!}{2^n(n!)^2}=\frac{1}{2^nn!}\frac{d^n(t^2-1)^n}{dt^n} \tag{13}$$

and

$$\int_{-1}^{1} P_n(t)P_m(t)\, dt = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases} \tag{14}$$

and normalized

$$\phi_n = \frac{y_n}{||y_n||} \tag{15}$$

Two functions, $A(x)$ and $B(x)$, are orthogonal in (a, b) if

$$\int_a^b A(x)B(x)dx = 0$$

A function, $A(x)$, is normalized, in (a, b), if

$$\int_a^b A^2(x)dx = 1$$

A set of functions $\{\phi_k(x)\}$, where $k \in Na$, is an orthonormal set in (a, b) if

$$\int_a^b \phi_m(x)\phi_n(x)dx = \delta_{mn}$$

The set $\{\psi_k(x)\}$ is orthonormal w.r.t. the density (or weight) function $w(x) \geq 0$ if

$$\int_a^b \psi_m(x)\psi_n(x)w(x)dx = \delta_{mn}$$

and the set $\phi_m(x) = \sqrt{w(x)} \cdot \phi_m(x)$ is an orthonormal set.

7 Gamma function

The Gamma function, convergent for $n > 0$, is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\, dx \tag{16}$$

or by the recurrence formula

$$\Gamma(n + 1) = n\Gamma(n) \tag{17}$$

where

$$\Gamma(1) = 1$$

In particular if n is apositive integer

$$\Gamma(n+1) = n! \qquad n \in \mathbf{N} \tag{18}$$

and it can be shown that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

For large n the following result may prove useful:

$$\Gamma(n+1) = n^n e^{-n} e^{\theta/12(n+1)} \sqrt{2\pi n} \qquad 0 < \theta < 1$$

and if n is an integer we can use Stirling's asymptotic formula for $n!$

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

Some results of the gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{sinx\pi}$$

the duplication formula

$$2^{2x-1}\Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\Gamma(2x)$$

and the beta function, defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

connected with the gamma function according to

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

8 Fourier integrals

Assuming $f(x)$ and $f'(x)$ piecewise continuous in every finite interval and $f(x)$ is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, then in $(-\infty, \infty)$ the Fourier integral theorem is

$$f(x) = \int_0^\infty \left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right)\, dx \tag{19}$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x \; d\alpha \tag{20}$$

where

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x \; dx \tag{21}$$

or equivalently

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u) \; du \; d\alpha \tag{22}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)} \; du \; d\alpha \tag{23}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x} \; d\alpha \int_{-\infty}^{\infty} f(u)e^{-i\alpha u} \; du \tag{24}$$

The functions can be simplified, if $f(x)$ is an odd function

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin\alpha x \; d\alpha \int_{0}^{\infty} f(u)\sin\alpha u \; du$$

and if $f(x)$ is even

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos\alpha x \; d\alpha \int_{0}^{\infty} f(u)\cos\alpha u \; du$$

9 Fourier Transforms

We call $F(\alpha)$ the Fourier transform of $f(x)$, and then $f(x)$ the inverse transform of $F(\alpha)$, or in symbols $F(\alpha) = \mathcal{F}\left(f(x)\right)$ and $f(x) = \mathcal{F}^{-1}\left(F(\alpha)\right)$ where

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)e^{-i\alpha u} \; du \tag{25}$$

and then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x} \; dx \tag{26}$$

if $f(x)$ is an odd function then we have the Fourier sine transform

$$F_S(\alpha) = \int_{0}^{\infty} f(u)\sin\alpha u \; du$$

and it follows that

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\alpha) \sin \alpha x \; d\alpha$$

similarly, if $f(x)$ is an even function

$$F_C(\alpha) = \int_0^\infty f(u) \cos \alpha u \; du$$

it then it follows that

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\alpha) \cos \alpha x \; d\alpha$$

If $F(\alpha)$ and $G(\alpha)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then we have

$$\int_{-\infty}^\infty f(x) g(x) \; dx = \frac{1}{2\pi} \int_{-\infty}^\infty F(\alpha) G^*(\alpha) \; d\alpha$$

and, in particular, if $f(x) = g(x)$ we then have Parseval's identity

$$\int_{-\infty}^\infty |f(x)|^2 \; dx = \frac{1}{2\pi} \int_{-\infty}^\infty |F(\alpha)|^2 \; d\alpha$$

and for sine transforms we have

$$\int_0^\infty f(x) g(x) \; dx = \frac{2}{\pi} \int_0^\infty F_S(\alpha) G_S(\alpha) \; d\alpha$$

and again, in particular, if $f(x) = g(x)$ we have

$$\int_0^\infty \left(f(x)\right)^2 \; dx = \frac{2}{\pi} \int_0^\infty \left(F_S(\alpha)\right)^2 \; d\alpha$$

and similarly for the cosine transform

$$\int_0^\infty f(x) g(x) \; dx = \frac{2}{\pi} \int_0^\infty F_C(\alpha) G_C(\alpha) \; d\alpha$$

and again

$$\int_0^\infty \left(f(x)\right)^2 \; dx = \frac{2}{\pi} \int_0^\infty \left(F_C(\alpha)\right)^2 \; d\alpha$$

10 Convolution theorem for Fourier transforms

The convolution of the functions $f(x)$ and $g(x)$ is commutative, associative and distributive and is defined by

$$f \star g = \int_{-\infty}^{\infty} f(u)g(x-u)\, du \tag{27}$$

and the convolution theorem is

$$\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g) \tag{28}$$

11 Bessel functions

Bessel functions arise as solutions of the equation called Bessel differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geq 0 \tag{29}$$

(for example from Laplace's equation $\nabla^2 u = 0$ expressed in cylindrical coordinates (ρ, ϕ, z). So by changing the variable x to λx, with λ constant, and obtaining the equation

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \tag{30}$$

we then find the general solution

$$y = c_1J_n(x) + c_2Y_n(x) \tag{31}$$

The solution $J_n(x)$, called Bessel function of the first kind of order n has a finite limit as x approaches 0 and is defined

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} + \dots\right)$$

or more succintly written as

$$J_n(x) = \sum_j \frac{(-1)^j(x/2)^{n+2j}}{j!\Gamma(n+j+1)} \tag{32}$$

while $Y_n(x)$ approaches ±∞ as x approaches zero.

For $n = 0$ we have

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots$$

For small x the asymptotic properties are:

$$J_n(x) \sim \begin{cases} 1 & n = 0 \\ \\ \frac{1}{2^n (n-1)!} & n > 0 \end{cases}$$

and

$$Y_n(x) \sim \begin{cases} \frac{2}{\pi} \ln x & n = 0 \\ \\ \frac{2^n (n-1)!}{\pi} x^{-n} & n > 0 \end{cases}$$

If n is an integer then $J_{-n}(x) = (-1)^n J_n(x)$ and if n is not an integer $J_n(x)$ and $J_{-n}(x)$ are linearly independent and the general solution for Bessel's differential equation is

$$y = AJ_n(x) + BJ_{-n}(x) \qquad n \neq 0, 1, 2 \dots$$

12 Legendre's differential equation

A function f is said to be analytic on an interval $(x_0 - r, x_0 + r)$ if f has a power-series expansion in this interval

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

convergent for $|x - x_0| < r$. If the coefficients of a homogeneous linear differential equation

$$y^{(n)} + P_1(x) y^{(n-1)} + \dots + P_n(x) y = 0$$

are analytic in an interval $(x_0 - r, x_0 + r)$ then there exist n indipendent solutions $u_1, \dots, u_n$ each of which is analytic on the same interval.

Legendre's differential equation

$$(1 - x^2) y'' + 2xy' + n(n+1) y = 0 \tag{33}$$

has power series solutions where n is any real constant. When n is any positive integer it has solutions called Legendre polynomials. This equation occurs for example from Laplace's equation $\nabla^2 u = 0$ expressed in sperical coordinates (r, θ, ϕ) with u indipendent of ϕ.

Let p and q be fixed functions in a space $C(a, b)$ of all real functions continuous on a closed interval $[a, b]$ and V the subspace consisting of all f which have a continuous second derivative in $[a, b]$ and which satisfy the boundary conditions

$$p(a)f(a) = 0 \qquad p(b)f(b) = 0$$

then the Sturm-Liouville operator $T : V \to C(a, b)$ is defined by the equation

$$T(f) = (pf')' + qf$$

The Legendre equation can be written as

$$\Big((x^2 - 1)y'\Big)' = n(n + 1)y \tag{34}$$

which has the form

$$T(y) = \lambda y$$

where T is a Sturm-Liouville operator, $T(f) = (pf')'$ with $p(x) = x^2 - 1$ and $\lambda = n(n + 1)$

The nonzero solutions of the Legendre equation are the eigenfunctions of T belonging to the eigenvalue $n(n + 1)$. Since $p(x)$ satisfies the boundary conditions $p(1) = p(-1) = 0$ the operator T is symmetric w.r.t. the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\, dx$$

the eigenfunctions belonging to distinct eigenvalues are orthogonal.

We divide through the Legendre equation by $1 - x^2$ and obtain

$$y'' + P_1(x)y' + P_2(x)y = 0 \tag{35}$$

where

$$P_1(x) = -\frac{2x}{1 - x^2} \qquad P_2 = \frac{n(n + 1)}{1 - x^2}$$

if $x^1 \neq 1$

For each n the nth degree polynomial solution is called the Legendre Polynomial of degree n

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{36}$$

where N equals the integral part of $n/2$, and the first Legendre polynomials are

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{3}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

We find the general series for y can be written as

$$y = c_1 P_n(x) + c_2 Q_n(x) \tag{37}$$