Hasse diagrams and n-dimensional space*

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With a few modifications to the rules for drawing Hasse diagrams for divisors we have a set of rules which allow one to draw the projection of n dimensional prisms onto the plane.

MSC 54B20, 51E24.

I would like to expand upon an interesting parallel between some Hasse diagrams and the projection of n-dimensional rectangles onto a plane.[1]

Given a *poset*,[2] we draw a direct line with the covering element above the one being covered. That is if, say, b covers a, then b is placed above a and

*set in LaTeX

[1] The n-dimensional cubes may also be referred to as hyper-cubes.

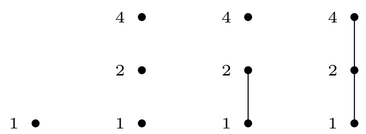
[2] Briefly reviewing the rules for drawing Hasse diagrams: A poset is a set with an ordering relation. If an element a is in an ordering relation with element b, $a \leq b$, or, in terms of products of sets, $(a, b) \in S \times S$ we then draw a connection between the two elements. This may be indirectly, if another element comes "between" them, otherwise (If

connected to it by a (possibly vertical) line.

$$\begin{matrix} b \\ | \\ a \end{matrix} \Leftrightarrow b \succ\!\!- a$$

To create the diagram of the divisors of n, first take all of the proper divisors of n and begin ordering them starting with 1. Take the smallest divisors (which will in fact be the primes) and draw a point for each of these above the point associated with the "1", Above each draw the next higher multiples of these, drawing a dot for each, and connecting them with each of their factors. Proceeding in this same way until all of the proper factors have been exhausted, we cap this with n itself, connecting with the larger factors.

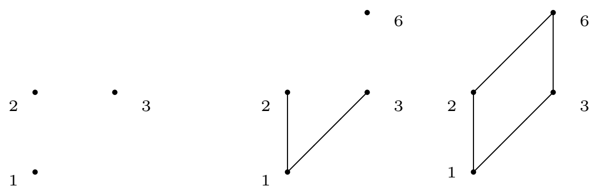
As a first example let us represent the divisors of 4 by its Hasse diagram. Shown are four successive steps: placing the "1", placing all the other factors and then connecting them.



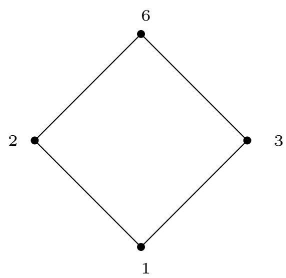
$\nexists x$ s.t. $a \leq x \leq b$ (where $\leq$ is the ordering relation then we say b *covers* a) we draw the connection directly.

In this case we have a *chain*.[3]

A bit more involved is the diagram for the divisors of 6. Note that since $1|2$ and $1|3$, trivially, we connect these, but that since 2 does not divide 3, these two cannot be connected. Then we have 6, which is divisible by both 2 and 3, and so it is placed above them and connected to them.



It does not take much to see that the diagram is topologically equivalent to a square.



Taking our cue from this, let us modify the rules somewhat.

1. To create the diagram of the divisors of n, first decompose n it into its prime factors, $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_m^{a_m}$.

2. Take the number of *different* prime factors, say r, and draw

[3]Every element is covered, or covers, another.

r axes coming from a point, the projection of each axis being at a different angle with respect to first selected axis.

3. If there are powers of a prime factor, we extend the line containing *that* prime factor, for each of the powers. We then draw parallels to the other axes at each "node" where the primes (or power of primes) lie, being careful never to extend a previously drawn segment (assuring that the procedure terminates).

4. At each intersection draw a node (It will be the multiple of the generating primes and one of the divisors of the given number n.

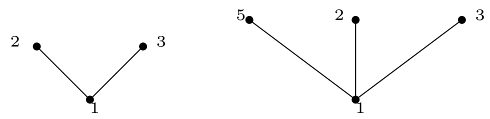
5. The algorithm ends when there are no more parallels that can be drawn from the last remaining node without extending some already existing segment.

For example, let us take the prime factors of a number, e.g., $24 = 2^3 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$ and let us look at the steps in the modified rules for a Hasse diagram.[4]

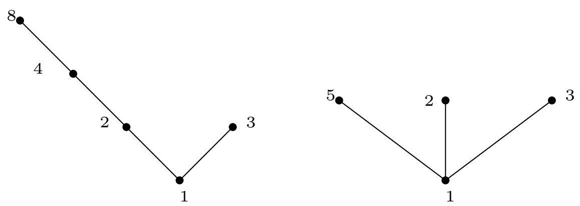
Having selected the points corresponding to the different primes, we draw

[4]For comparison, a Hasse diagram would be:

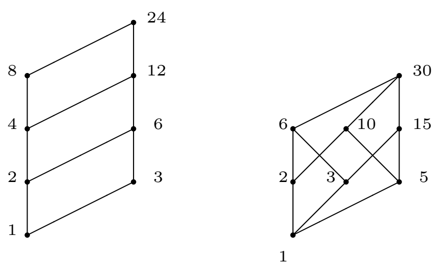
the axes from “1” to the points:

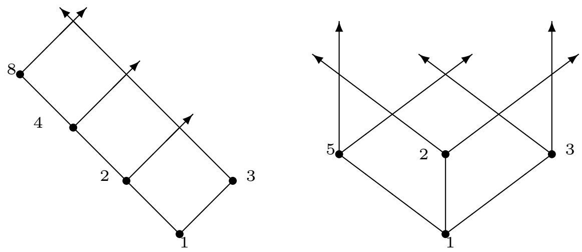


Then continuing as described above, we extend the axes containing the power of primes:

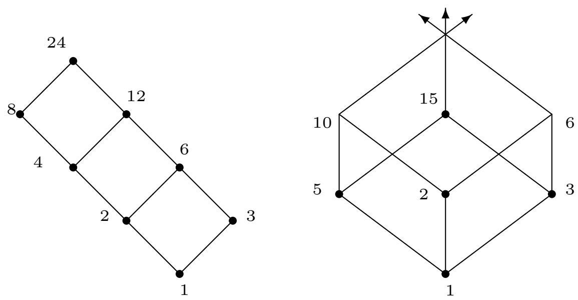


We now draw the parallels to the other axes and find the nodes where they meet:

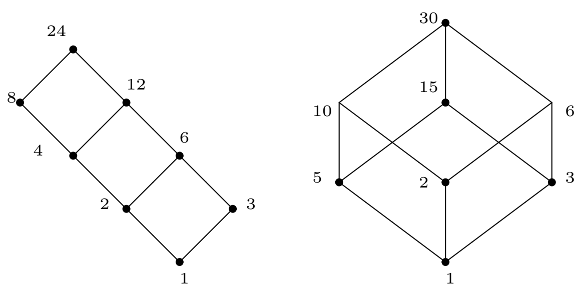




If all the divisors have not been exhausted we can draw still more parallels from the new nodes:



We now complete the second diagram:

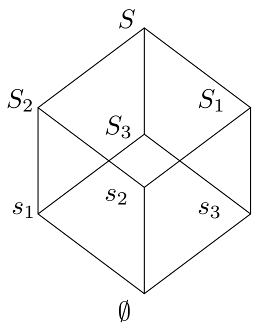


By associating a (unique) prime factor for each axis we have effectively "projected" 1, 2 and 3 dimensional rectangles onto the plane. The diagram for the divisors of 30 is in fact a projection of a 3-d cube onto a plane.[5]

Note that for numbers having less than three distinct prime divisors the diagram will always be a plane, so that for the ordering relation "divides" we have a trivial way of finding which numbers generate planar graphs.

Proceeding along similar lines we can draw the diagram for the divisors of 90, remembering to extend the "3" axis since there is the factor 3^2. In a more general notation we are using a number of the form $n = p_1 \cdot p_2 \cdot p_3^2$. The three axes are then formed by p_1, p_2 and p_3^2. Note that p_3 and p_3^2 are multipled by p_1 and p_2,[6]

[5]This is identical, by the way, to the Hasse diagram for the power set of S, $\mathcal{P}(S)$, and the relation "subset of", that is, for the poset $\langle \mathcal{P}, \subseteq \rangle$:

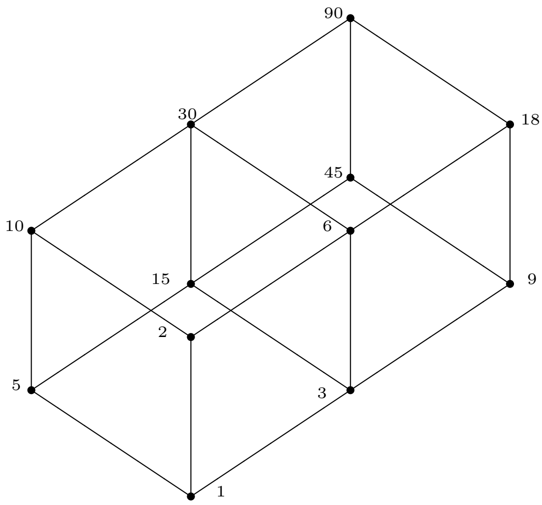


$S = \{a, b, c\}$

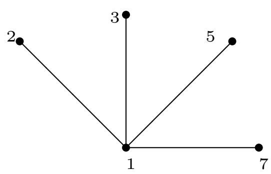
$S_1 = \{a, b\}$ $S_2 = \{a, c\}$
$S_3 = \{b, c\}$

$s_2 = \{b\}$
$s_1 = \{a\}$ $s_3 = \{c\}$

[6]In the example: 3 and 2 are multiplied to get the value of the vertex "6", and the 9 and 2 are multiplied to get "18", we do *not* multiply 9 by 6.

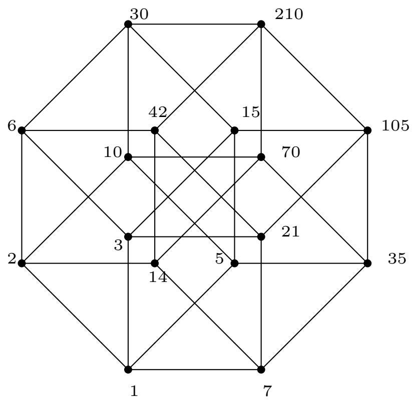


So far we have seen examples of diagrams for numbers of the form $n = p_1^r \cdot p_2^s \cdot p_3^t$ for $r, s, t \in \{0, 1, 2, 3\}$. For dimensions higher than 3 we can still imagine that lines at different angles from the same node be projections of different axes.[7] As an illustration let us look at a four dimensional cube. Starting with a number composed of 4 prime factors,i.e. $n = p_1 \cdot p_2 \cdot p_3 \cdot p_4$, such as "210", we now draw a modified Hasse diagram.

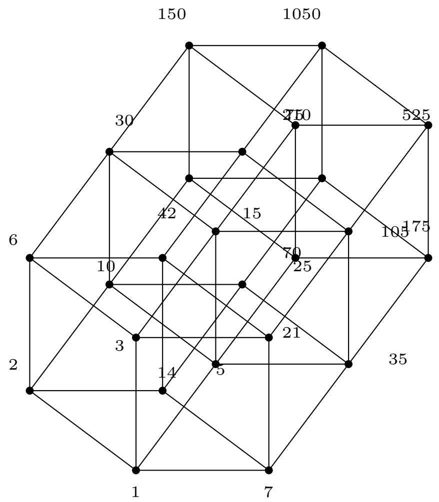


[7]A practical note: the angles and lengths should be selected so that the ends of line line segments do not fall along other line segments. It is best not to use segments of the same length as they will form nodes that will fall along some other segments, sometimes it takes a bit of experimenting to obtain the right proportions.

It is quite instructive to proceed according to the rules, that is drawing the three parallels from each of the “prime” nodes and labeling the points of intersection with the product of the “generating” primes. Then drawing the two parallels from each of these, and one remaining parallel from their intersection. We will then obtain the finished diagram.



This, of course can be extended, for example, to a 4^{th} dimensional (rectangular) prism, using the diagram for the divisors of a number of the form $n = p_1^2 \cdot p_2 \cdot p_3 \cdot p_4$ (for example $1500 = 2 \cdot 3 \cdot 5^2 \cdot 7$).



Extending a bit further (without labeling the nodes) we can draw the projection of a 5^{th} dimensional prism.